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Author(s)	Kishimoto, Akitaka
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Compact group actions on C*-algebras

東北大教養 岸本晶孝(Akitaka Kishimoto)

1. Introduction

The general problem I am concerned with is as follows (cf.[12-15]): For a given C*-dynamical system (A, G, α), analyze $(\hat{A}, G, \alpha^*)$, where A is a C*-algebra with its dual $\hat{A}$, G is a locally compact group, and α is a continuous action of G on A by automorphisms. Here I do not mind assuming A is simple; a bit more reluctantly assuming A is separable. (In fact for most of the relevant results so far obtained we seem to have to assume, at least, that A is prime and separable.) A prototype C*-dynamical system I am thinking of in this study is an infinite tensor product type action on a UHF algebra or more generally a quasi-free action on a CAR algebra, which seems to be endowed with many of the physical properties in the real world.

In [13] and [15] I defined types of orbits in $\hat{A}$ under α^*. Namely, for $\pi \in \hat{A}$ regarded as an irreducible representation on some Hilbert space, say H_π, one constructs a representation $\tilde{\pi}$ of A by

$$\tilde{\pi} = \int_G^{\oplus} \pi\circ\alpha_t \, dt$$

on $L^2(G, H_\pi)$, and define the type of the orbit through π under α^* to be the type of $\tilde{\pi}(A)''$ as a von Neumann algebra. Since it is easily shown that $\tilde{\pi}(A)''$ is homogeneous, the orbit type is especially either of type I, II, or III. I am most interested in exploring type III orbits (as this seems to be or have been the most natural attitude toward everything named type III), but without any results worth mentioning. As usual, the type I case is the most manageable.

I should mention another approach to the C*-dynamical systems in general --- the one taken by e.g., [16], of exploring the Connes spectrum (cf. [17]) or a Connes spectrum to be. I tried this in [15] perhaps too consciously. It now seems that the notion of Connes spectrum with an additional condition, e.g., the existence of a covariant irreducible representation is good enough. Along these lines (as it happened) the results in [16] was cultivated in [2], where the group G is assumed to be compact and abelian.

In this expository note I am mainly concerned with the case G is compact (and non-abelian) and describe some results (envolving type I orbits) obtained in [3]. I quote from there:

1.1. Theorem. Let A be a separable C*-algebra, G a compact group with $G \neq \{e\}$, and α a faithful continuous action of G on A. Then the following conditions are equivalent:

(i) There exists a faithful irreducible representation π of A such that $\pi | A^{\alpha}$ is irreducible.

(ii) There exists a pure invariant state ω of A such that the GNS representation $\pi_{\omega|A^{\alpha}}$ of A^{α} is faithful.

(iii) Let $\{\xi_n\}$ be an arbitrary sequence of finite-dimensional unitary matrix representation of G, and let β be the infinite tensor product action $\bigotimes_{n=1}^{\infty} \mathrm{Ad}\,\xi_n$ of G on the UHF algebra $C = \bigotimes_{n=1}^{\infty} M_{d_n}$, where d_n is the dimension of ξ_n and M_{d_n} is the $d_n \times d_n$ matrix algebra. It follows that there exists a globally invariant C*-subalgebra B of A, and a closed α^{**}-invariant projection q of A such that (a) $q \in B'$, (b) $qAq = Bq$, (c) $q \in I^{**}$ ($\subset A^{**}$) for any non-zero closed ideal I of A, and (d) the C*-dynamical system $(Bq, G, \alpha^{**}|Bq)$ is isomorphic to (C, G, β).

(iv) For each $\gamma \in \hat{G}$ there exists a $\delta_\gamma > 0$ such that for each unit vector $\lambda \in \mathbb{C}^d$ with $d = \dim \gamma$, there is a central sequence $\{y_n\}$ in $\{x\cdot\lambda \; : \; x \in A_1^\alpha(u^\gamma), \; \|x\cdot\lambda\| = 1\}$ with

$$\limsup \|ay_n\| \geq \delta_\gamma \|a\|, \quad a \in A,$$

where u^γ is a fixed unitary matrix representation of G in class γ.

(v) Condition (iv) holds with $\delta_\gamma = 1$, $\gamma \in \hat{G}$.

Note that the orbit through $\pi \in \hat{A}$ as in (i) is of type I; in this case the center of $\tilde{\pi}(A)''$ is isomorphic to $L^\infty(G)$ (together with natural actions of G). Also note that the orbit through $\pi_\omega (\in \hat{A})$ with ω as in (ii) is of type I; in this case π_ω is fixed under α_t^*, $t \in G$. Furthermore studying C*-dynamical systems like (C, G, β) in (iii) would yield many orbits in $\hat{A}$ of various types.

In Section 2 we discuss orbit types in details and give a characterization of type I orbits in the case G is a (non-abelian) locally compact group (a slight generalization of a result in [15]).

In Section 3 we discuss, roughly speaking, the problem of when a C*-algebra is weakly dense in a larger C*-algebra in some irreducible representation. In fact this is quite essential in generalizing some of the results in [2] to the case G is non-abelian in our treatment. More preferably we should disregard this; to do so we would need a certain characterization of 'properly outer' endomorphisms, generalizing the corresponding notion of automorphisms.

In Section 4 we discuss invariant Hilbert spaces for a C*-dynamical system (with G compact). We will discuss more or less throughly a part of 4.2, which is not really required for the proof of 1.1, only to explain a general idea and to supplement a result in [9]. The implication (ii) $\Rightarrow$ (iii) in 1.1 follows from 4.2 (part with no proof ---see [9]) and [5].

In Section 5 we discuss endomrphisms, which are the dual objects of a compact action. The implication (i) $\Rightarrow$ (ii) in 1.1, which is the hardest in this theorem, follows from 5.7, 5.9, and 5.10 (where 5.9 relies on Section 3 mentioned above).

In Section 6 we give a brief discussion on infinite tensor product type actions on UHF algebras; that (iii) implies (i) would easily follow from this kind of discussion.

In Section 7 we discuss central sequences, proving that (i) $\Rightarrow$ (v) in 1.1. Note that (v) $\Rightarrow$ (iv) is trivial. We will not give a proof to the implication (iv) $\Rightarrow$ (i). But in Section 8 we shall instead discuss the asymptotic abelianess condition, which is certainly stronger than (iv), under which a representation as in (i) is constructed without using the compactness assumption on the group G.

In Section 9 we give another type of condition which is equivalent to the ones in 1.1, i.e., a condition which could be placed between (i) and (ii) in character, see 9.1.

Finally we remark that (ii) in 1.1 could be replaced by

(ii') There exists a family $\{\omega_i\}$ of pure invariant states of A such that $\oplus \pi_{\omega_i | A^\alpha}$ is a faithful representation of A^α.

It is not hard to prove that (ii') implies (iii) (see 2.1 in [5], 2.1 in [9], and 4.5).

I would like to thank Professor Sakai for his advices at an early stage of this study and Professor Nakagami for encouraging me to write [13] which was essentially the starting point for this kind of work.

2. Orbit types

Let (A, G, α) be a C*-dynamical system as in section 1. Let π be an irreducible representation on a Hilbert space H_π. Let $K(G, H_\pi)$ be the linear space of H_π-valued continuous functions on G with compact support and define an inner product on $K(G, H_\pi)$ by

$$(\xi, \eta) = \int_G (\xi(t), \eta(t))\, d\mu(t)$$

for $\xi, \eta \in K(G, H_\pi)$, where $d\mu$ is a right Haar measure on G. For each $a \in A$ define $\tilde{\pi}(a)$ on $K(G, H_\pi)$ by

$$(\tilde{\pi}(a)\xi)(t) = \pi\circ\alpha_t(a)\,\xi(t)$$

for $\xi \in K(G, H_\pi)$. Then it is obvious that $\tilde{\pi}$ defines a bounded *-representation of A and hence extends to a representation of A on the completion of $K(G, H_\pi)$, i.e., $L^2(G, H_\pi)$.

Define a unitary representation U of G on $L^2(G, H_\pi)$ by right translations, i.e.,

$$(U(t)\xi)(s) = \xi(st),$$

for $\xi \in K(G, H_\pi)$. Then $\tilde{\pi}$ is a covariant representation with this U, i.e., $U(t)\tilde{\pi}(a)U(t)^* = \tilde{\pi}\circ\alpha_t(a)$, $a \in A$, $t \in G$. Denote by $\bar{\alpha}$ the action of G on $\mathcal{M} = \tilde{\pi}(A)''$ implemented by U.

<u>2.1. Lemma</u>. Let $Q \in \mathcal{M}$ and $h \in K(G) \equiv K(G, \mathbf{C})$. Define

$$\bar{\alpha}_h(Q) = \int_G h(t)\,\bar{\alpha}_t(Q)\, d\mu(t),$$

and $\alpha_h(x)$ similarly for $x \in A$. Let $\{x_\mu\}$ be a bounded net in A such that $\tilde{\pi}(x_\mu)$ converges weakly to Q. Then $\pi(\alpha_h(x_\mu))$ converges weakly to $\bar{\alpha}_h(Q)$, $\pi\circ\alpha_t(\alpha_h(x_\mu))$ converges weakly, say to T_t, and

$$\bar{\alpha}_h(Q) = \int_G^{\oplus} T_t\, d\mu(t).$$

Proof. Let $y_\mu = \alpha_h(x_\mu)$ and $T = \tilde{\alpha}_h(Q)$. We first claim that $\tilde{\pi}(y_\mu)$ converges weakly to T. Let $\xi, \eta \in K(G, H_\pi)$ and choose a subsequence $\{\mu_n\}$ such that $(\tilde{\pi}(y_{\mu_n})\xi, \eta)$ converges. Since the closed linear span of $\{u(t)^*\xi, u(t)^*\eta ; t \in \operatorname{supp} h\}$ is separable, there is in turn a subsequence $\{\nu_n\}$ of $\{\mu_n\}$ such that

$$(\pi(x_{\nu_n})u(t)^*\xi, u(t)^*\eta) \to (Qu(t)^*\xi, u(t)^*\eta), \quad t \in \operatorname{supp} h$$

as n goes to infinity. Hence $(\tilde{\pi}(y_{\nu_n})\xi, \eta)$ converges to $(T\xi, \eta)$, which implies that $\tilde{\pi}(y_\mu)$ converges to T.

Since the functions $t \mapsto \alpha_t(y_\mu)$ are equi-continuous (in norm), it follows that for any compact subset K of G and any separable subspace H of H_π, there is a subsequence $\{y_{\mu_k}\}$ such that $\pi \circ \alpha_t(y_{\mu_k})$ converges weakly, say to T_t, on H for each $t \in K$. Note that $t \to T_t$ is norm-continuous. Thus if $K \supset \operatorname{supp} \xi$, and $\xi(t), \eta(t) \in H$ for all $t \in G$, then

$$\int (T_t\,\xi(t), \eta(t))\,d\mu(t) = (T\xi, \eta).$$

By taking various K and H, one can conclude that there is a continuous bounded function $T(\cdot)$ of G into $B(H_\pi)$ such that

$$\int (T(t)\,\xi(t), \eta(t))\,d\mu(t) = (T\xi, \eta)$$

for $\xi, \eta \in K(G, H_\pi)$. And it easily follows that $\pi \circ \alpha_t(y_\mu)$ converges to $T(t)$.

It follows that the center Z of $\mathcal{N}$ is contained in $L^\infty(G) \otimes \mathbb{C}1$. Thus $\tilde{\alpha}$ is ergodic on Z, i.e., $Z^{\tilde{\alpha}} = \mathbb{C}1$. Hence $\mathcal{N}$ is homogeneous as is claimed in section 1, i.e., for any two non-zero central projections e and f, $\mathcal{N}e$ and $\mathcal{N}f$ are mutually isomorphic.

Since Z is globally $\tilde{\alpha}$-invariant, Z can be identified with $L^\infty(H\backslash G)$

for some closed subgroup H of G. For example let C be the set of $Q \in Z$ such that $t \mapsto \bar{\alpha}_t(Q)$ is norm-continuous. Then H can be defined as $\{h \in H;\ f(h) = f(e),\ f \in C\}$, where e is the identity element.

2.2. Proposition. Suppose that the C*-algebra A is separable and the locally compact group G has a countable basis. Let $\pi \in \hat{A}$ and define $G_\pi = \{t \in G;\ \pi\circ\alpha_t = \pi\}$. Then the following conditions are equivalent:

(i) $\tilde{\pi}(A)''$ is of type I.

(ii) G_π is closed and $\tilde{\pi}(A)'' \cap \tilde{\pi}(A)' = L^\infty(G_\pi\backslash G) \otimes \mathbb{C}1$.

Proof. Suppose (i) and let C be the C*-subalgebra of the center Z as above. Let $Q \in C$ and $h \in K(G)$. Choose a bounded net $\{x_\nu\}$ in A such that $\tilde{\pi}(x_\nu)$ converges weakly to Q. Then by Lemma 2.1 $\pi\circ\alpha_t(\alpha_h(x_\nu))$ converges to a multiple of the identity for any $t \in G$. If $t \in G_\pi$, one must have that $\bar{\alpha}_h(Q)(t) = \bar{\alpha}_h(Q)(t)$ i.e., $Q(t) = Q(e)$ for all $Q \in C$. Thus G_π is contained in H where H is defined by the property that $Z \simeq L^\infty(H\backslash G)$.

Let $d\mu_1$ be a quasi-invariant measure on $H\backslash G$ and let f be a measurable function of $H\backslash G$ into G such that $t = Hf(t)$. Then

$$\int_G^\oplus \pi\circ\alpha_t\, d\mu(t) \simeq \int_{H\backslash G}^\oplus \left\{ \int_H^\oplus \pi\circ\alpha_{sf(t)}\, ds \right\} d\mu_1(t)$$

where ds is a right Haar measure on H (the measure on G defined by $ds \times d\mu_1$ via f is equivalent to the Haar measure $d\mu$ on G). Since the integral over $H\backslash G$ is central and the weak closure of A in each integrand representation is isomorphic with each other, one can conclude that

$$\left\{ \int_G^\oplus \pi\circ\alpha_t(A)\, d\mu(t) \right\}'' \simeq L^\infty(H\backslash G) \otimes \left\{ \int_H^\oplus \pi\circ\alpha_s(A)\, ds \right\}''.$$

Thus the direct integral of $\pi\circ\alpha_s$ over H is of type I factor and hence α_s is weakly inner in this representation for all $s \in H$. Therefore one can conclude that $H \subset G_\pi$.

The proof of the converse is similar to the above.

From condition (ii) it is not difficult to see if there are non-type I orbits: Often the stabilizer G_π is not closed. But it remains hard to produce non-type I orbits where the stabilizer is trivial.

Finally we note the following result:

2.3. Proposition. Let A be a separable prime C*-algebra and let α be a faithful continuous action of a compact group $G \neq \{e\}$. Let π be a faithful irreducible representation. Then the following conditions are equivalent:

(i) $\pi | A^\alpha$ is irreducible.

(ii) $\tilde{\pi}(A)'' \cap \tilde{\pi}(A)' = L^\infty(G) \otimes \mathbb{C}1$.

Furthermore in this case A^α is prime and has no minimal projections.

The proof is straightforward (cf. [15] and Section 9).

3. Weak density

Let A be a C*-algebra and B a C*-subalgebra of A. We consider the problem of when B is weakly dense in A in some representation.

3.1. Proposition. [3] Take a pair A, B as above and suppose that A is separable. Then the following conditions are equivalent:

(i) There exists a $\delta > 0$ such that for any $x, y \in A$

$$\sup\{\|xby\| : b \in B, \|b\| \leq 1\} \geq \delta \sup\{\|xay\| : a \in A, \|a\| \leq 1\}.$$

(ii) Condition (i) holds with $\delta = 1$.

(iii) There exists a faithful family of irreducible representations of A whose restrictions to B are also irreducible.

(iv) For any decreasing sequence $\{I_n\}$ of non-zero ideals of A such that I_m is essential in I_n for $m > n$, there is an irreducible representation π of A such that $\pi|I_n \neq 0$ for any n and $\pi|B$ is irreducible.

Let T be the set of $e \in A$ with $e \geq 0$, $\|e\| = 1$ satisfying

$$H(e) \equiv \{a \in A : ea = ae = a\} \neq \{0\}.$$

It easily follows that Condition (i) remains equivalent if the inequalities are only required for $x, y \in T$. Note that if A is prime, then

$$\sup\{\|xay\| : a \in A, \|a\| = 1\} = \|x\|\|y\|.$$

Since the implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are rather obvious, it suffices to prove that (i) implies (iv).

Proof of (i) $\Rightarrow$ (iv). Let $\{I_n\}$ be a sequence as in (iv) and let $\{u_n\}$ be a dense sequence in the unitaries of A (or $A + \mathbb{C}1$ if $A \not\ni 1$). We enumerate $\{(u_k, u_m) : k, m = 1, 2, \ldots\}$ and let $\{(u_n, v_n)\}$ be the resulting sequence.

Fix $e_1 \in T \cap I_1$. We choose sequences $e_n \in T \cap I_n$ ($n \geq 2$), $a_n \in T \cap H(e_n)$, and $b_n \in B_1 \equiv \{ b \in B : \|b\| \leq 1 \}$, satisfying the following conditions:

$$\sup \mathrm{Spec}(y_n) > \lambda_n - \delta/2n,$$

where

$$\lambda_n = \sup \{ \| p_n(u_n b v_n + v_n^* b^* u_n) p_n \| : b \in B_1 \},$$

$$y_n = a_n(u_n b_n v_n + v_n^* b_n^* u_n^*) a_n,$$

p_n is the open projection corresponding to $H(e_n)$, and e_n is chosen from $T \cap B(f_{n-1}(y_{n-1})) \cap I_n$ for $n \geq 2$ with $f_n(t) = f((\lambda_n - \delta/2n)^{-1} t)$, where

$$f(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 1 \\ 1 & t \geq 1 . \end{cases}$$

Note that $\lambda_n \geq \delta$ (by condition (i)) and that the arguments here are much the same as in [15].

Proving existence of those sequences, let f be a pure state of A such that $f(e_n) = 1$ for all n. Then $\pi_f \mid I_n \neq 0$ for all n. Since $f(a_n) \to 1$ as $n \to \infty$, it follows that for any unitaries u, v of A (or $A + \mathbb{C}1$), there is a $Q \in \pi_f(B)''$ such that $\|Q\| \leq 1$, and

$$\mathrm{Re} \langle Q \pi_f(u) \Omega_f , \pi_f(v) \Omega_f \rangle \geq \delta/2 .$$

Thus, by using Kadison's transitivity theorem, one can conclude that $\pi_f(B)'' = B(H_f)$.

It is not clear whether (i) in this proposition is equivalent to:
(i') If $xAy \neq \{0\}$ with $x, y \in A$, then $xBy \neq \{0\}$. I do not know whether this is true or not even if B is further supposed to be a hereditary C*-subalgebra of A. See [2] for relevant results.

4. Invariant Hilbert spaces

In this section we assume that the group G is compact. One problem associated with the C*-dynamical system (A, G, α) is whether there are sufficiently many α-invariant Hilbert spaces in A (see e.g. [6 ~ 8]). First we give:

4.1. Definition. A subspace $\mathcal{H}$ in the C*-algebra A is called a Hilbert space if there is a non-zero positive $a \in A$ such that $y^*x \in \mathbb{C}a$ for any $x, y \in \mathcal{H}$ and $\mathcal{H}$ is a Hilbert space with the inner product $(\ , \ )$ defined by $(x, y)a = y^*x$.

Note that the inner product is unique up to a positive constant multiple. If we can choose a to be a projection, we usually do so.

Actually we are only concerned with finite-dimensional Hilbert spaces, and from now on we assume that Hilbert spaces (in a C*-algebra) are always finite-dimensional.

If H is an invariant Hilbert space, then for $x, y \in H$ and a as in 4.1,

$$(\alpha_g(x), \alpha_g(y))a = \alpha_g(y^*x) = (x, y)\,\alpha_g(a).$$

Hence $a \in A^{\alpha}$ and $(\alpha_g(x), \alpha_g(y)) = (x, y)$. Thus the α restricted to H defines a unitary representation of G.

Let u be a unitary matrix representation of G. and let d be the dimension of u, $\dim u$. For each $n = 1,2,\ldots$ define

$$A_n^{\alpha}(u) = \{ x \in A \otimes M_{nd} : \alpha_g(x) = xu(g), \ g \in G \}$$

where M_{nd} is the linear space of $n \times d$ matrices and $\alpha_g(x) = (\alpha_g(x_{ij}))$ for $x = (x_{ij})$.

If there is a non-zero $x \in A_1^{\alpha}(u)$ such that $x^*x = a \otimes 1 \in A \otimes M_d$ for some $a \in A$, then the subspace H spanned by the components $x_1, \ldots x_d$

of x is an invariant Hilbert space and $\alpha | H$ defines u. Note that if u is irreducible, the requirement $x^*x = a \otimes 1$ is equivalent to the one that $\alpha_g(x^*x) = x^*x$ or $x^*x \in A^\alpha \otimes M_d$.

We give one result concerning invariant Hilbert spaces.

4.2. Theorem. [9] Let A be a separable prime C*-algebra and let α be a faithful continuous action of a compact group G on A with $G \neq \{e\}$. Suppose one of the following two conditions:

(i) There exists a faithful irreducible representation π of A such that $\pi | A^\alpha$ is irreducible.

(ii) There exists an invariant pure state ω of A such that $\pi_\omega | A^\alpha$ is faithful.

Then A^α is prime and has no minimal projections, and for any unitary matrix representation u of G and for any $e \in T \cap A^\alpha$ there is an $x \in A_1^\alpha(u)$ such that $ex = x$, and $x^*x \in A^\alpha \otimes 1$.

Note that (i) and (ii) are actually equivalent with each other (if 1.1 is proved).

Before we give a proof under condition (i), we present:

4.3. Lemma. [3] Let φ be a pure state of A such that $\pi_\varphi | A^\alpha$ is irreducible. Let ψ be a pure state of A such that $\psi | A^\alpha = \varphi | A^\alpha$. Then there exists a $g \in G$ such that $\psi = \varphi \circ \alpha_g$.

Proof. For a continuous non-negative function f on G, let

$$\psi_f = \int_G f(g)\, \varphi \circ \alpha_g \, dg$$

Then $\psi_f \leq c \int_G \varphi \circ \alpha_g \, dg$ with $c = \max\{f(g) : g \in G\}$, since

$$\psi_f + \psi_{c-f} = \psi_c = c \int_G \varphi \circ \alpha_g \, dg$$

As

$$\int_G^{\oplus} \pi_\varphi \circ \alpha_g \, dg$$

is a central and irreducible decomposition, there is a bounded non-negative measurable function h on G such that

$$\psi_f = \int_G h(g)\, \varphi \circ \alpha_g \, dg .$$

Hence if $\int_G f(g)\, dg = 1$, ψ_f is in the weak closure of the convex hull of $\varphi \circ \alpha_g$, $g \in G$, and thus ψ is too. Since the extreme points of this convex set is $\{ \varphi \circ \alpha_g , g \in G \}$ which is closed and ψ is pure, it follows that $\psi = \varphi \circ \alpha_g$ for some $g \in G$.

Now we come to the proof with (i) in 4.2. The first part is trivial (as we remarked in 2.2).

Let $e \in T \cap A^\alpha$ and $H(e) = \{ a \in A^\alpha : ae = ea = a \}$ as before. Fix an irreducible unitary matrix representation u of G and let B be the closed linear span of $x^* A \otimes M_d\, y$ with $x, y \in H(e) A_1^\alpha(u)$. Then B is a non-zero hereditary C*-subalgebra of $A \otimes M_d$ satisfying $A^\alpha B A^\alpha \subset B$, with $A^\alpha = A^\alpha \otimes 1$.

We first prove, by using (i),

(I) $B \cap A^\alpha \neq \{0\}$.

Then by routine arguments we can show

(II) There exists a non-zero $x \in A_n^\alpha(u)$ for some $n = 1,2,\ldots$ such that $ex = x$ and $x^* x \in A^\alpha \otimes 1$.

Again by using (i) we prove

(III) There exists a non-zero $x \in A_n^\alpha(u)$ for some $n = 1,2,\ldots, d$ such that $ex = x$ and $x^* x \in A^\alpha \otimes 1$.

Finally by using the fact that A^α has no minimal projections one can show

(IV) There exists a non-zero $x \in A_1^\alpha(u)$ such that $ex = x$ and $x^* x \in A^\alpha \otimes 1$.

Once this is obtained it is easy to prove the results for arbitrary unitary matrix representations of G.

Proof of (I). Contrarily suppose that $B \cap A^{\alpha} = \{0\}$. Then it follows that for any state $\hat{\varphi}$ of A^{α} there is a state $\tilde{\varphi}$ of $A \otimes M_d$ such that $\tilde{\hat{\varphi}} | A^{\alpha} = \hat{\varphi}$ and $\tilde{\hat{\varphi}} | B = 0$.

Take a unit vector $\xi \in \mathcal{H}_{\pi}$ with π as in (i), and define a pure state φ of A by

$$\varphi(x) = (\pi(x)\xi, \xi), \quad x \in A.$$

Any pure state extension of $\varphi | A^{\alpha}$ to $A \otimes M_d$ is of the form $\varphi \circ \alpha_g \otimes \omega$ such that $g \in G$ and ω is a pure state of M, by the previous lemma. Then in the GNS representation π_1 associated with this extension, the support projection of $\pi_1(B)''$ is of the form $1 \otimes e$ with e a projection of M_d since $\pi_1(A^{\alpha} \otimes 1)' \cong M_d$. But since for any non-zero $x \in H(e) A_1^{\alpha}(u)$ and $\lambda \in \mathbb{C}_j^d$

$$\sum_{i,j} \bar{\lambda}_i x_i^* x_j \lambda_j = (\sum \lambda_i x_i)^* (\sum \lambda_i x_i) \neq 0,$$

we must have that $e = 1$. Thus $\|\tilde{\varphi} | B\| = 1$ for any pure state extension $\tilde{\varphi}$ of $\varphi | A^{\alpha}$, and hence for any state extension of $\varphi | A^{\alpha}$. Since this is a contradiction, one must have that $B \cap A^{\alpha} \neq \{0\}$.

We omit the (easy) proof of (II) and refer to [9].

To proove (III) we need

<u>4.4. Lemma.</u>[3] Let φ be a pure state of A such that $\pi_{\varphi} | A^{\alpha}$ is irreducible. Let $\{z_k\}$ be a sequence in $T \cap A^{\alpha}$ such that $z_k z_{k+1} = z_{k+1}$, $k = 1, 2, \ldots$ and the limit of $\{z_k\}$ in $(A^{\alpha})^{**}$ is the support projection of $\varphi | A^{\alpha}$ [18]. Then for any $x \in A \otimes M_d$ it follows that

$$\lim \|z_k x z_k\| = \sup \{\|R_{\varphi \circ \alpha_g}(x)\| : g \in G\}$$

where R_{ψ} is the map of $A \otimes M_d$ into M_d defined by $R_{\psi}(x) = (\psi(x_{ij}))$ for $x = (x_{ij})$.

Proof. Since $\|R_{\varphi \circ \alpha_g}\| = 1$ and $\varphi \circ \alpha_g(z_k) = 1$, it follows that

$$\|R_{\varphi \circ \alpha_g}(x)\| = \|R_{\varphi \circ \alpha_g}(z_k x z_k)\| \leq \|z_k x z_k\|.$$

Since $\|z_k x z_k\| \geq \|z_{k+1} x z_{k+1}\|$, the limit of $\|z_k x z_k\|$ exists, say λ, and it follows that

$$\lambda \geq \sup \{ \|R_{\varphi \circ \alpha_g}(x)\| : g \in G \}.$$

On the other hand for k and m,

$$\lambda^2 \leq \|z_k x z_m x^* z_k\|.$$

Since $z_k x z_m x^* z_k$ is decreasing in $A \otimes M_d$ as m goes to infinity, one obtains that for any k,

$$\lambda^2 \leq \|z_k x p x^* z_k\|$$

where $p = \lim z_k$ in A^{**}. Since there is a pure state ω of $A \otimes M_d$ such that $\omega(z_k x p x^* z_k) = \|z_k x p x^* z_k\|$, one has for $\pi = \pi_\omega$ that $\|\pi(z_k x p x^* z_k)\| = \|z_k x p x^* z_k\| = \|p x^* z_k^2 x p\|$. In particular $\pi(p) \neq 0$. Since π is irreducible, it follows from 4.3 that π is equivalent to $\pi_{\varphi \circ \alpha_g} \otimes id$ where id is the identity representation of M. Hence

$$\|z_k x p x^* z_k\| = \|\pi(p x^* z_k^2 x p)\| = \|R_{\varphi \circ \alpha_g}(x^* z_k^2 x)\| \leq \|R_{\varphi \circ \alpha_g}(x^* z_k x)\|.$$

Thus one obtains that for any k,

$$\lambda^2 \leq \sup \{ \|R_{\varphi \circ \alpha_g}(x^* z_k x)\| : g \in G \}.$$

Since $\|R_{\varphi \circ \alpha_g}(x^* z_k x)\| = \|R_\varphi(\alpha_g(x^*) z_k \alpha_g(x))\|$ is equi-continuous as functions in $g \in G$, and G is compact, it follows that

$$\lim_{k \to \infty} \sup_{g \in G} \|R_{\varphi \circ \alpha_g}(x^* z_k x)\| = \sup_{g \in G} \lim_{k \to \infty} \|R_{\varphi \circ \alpha_g}(x^* z_k x)\|$$

where sup is really max. On the other hand

$$\lim \|R_{\varphi \circ \alpha_g}(x^* z_k x)\| = \|R_{\varphi \circ \alpha_g}(x)\|^2$$

because for $a, b \in A$,

$$\lim \varphi \circ \alpha_g(a z_k b) = \varphi \circ \alpha_g(a) \varphi \circ \alpha_g(b).$$

Thus one obtains that

$$\lambda^2 \leq \sup \{ \| R_{\varphi\circ\alpha_g}(x) \|^2 : g \in G \},$$

which completes the proof.

Proof of (III). Let m be the smallest positive integer for which there exists a non-zero $x \in A_m^\alpha(u)$ such that $ex = x$ and $x^*x = a \otimes 1 \in A^\alpha \otimes M_d$. We may suppose that $a \in T$.

Let ξ be a unit vector of $[H(a)H_\pi]$ with π as in (i) and let φ be the associated vector state of A. Noting that $\varphi | A^\alpha$ is pure, we let $\{z_k\}$ be a decreasing sequence in $T \cap A^\alpha$ such that $z_1 = a$, $z_k z_{k+1} = z_{k+1}$, and the limit of z_k in $(A^\alpha)^{**}$ is the support projection of $\varphi | A^\alpha$.

Denote by x_i the i-th row of x. Then $x_i \in A_1^\alpha(u)$ and

$$x^*x = \sum_{i=1}^{m} x_i^* x_i .$$

Since $R_{\varphi\circ\alpha_g}(x_i^*x_i) = u(g)^* R_\varphi(x_i^*x_i)u(g)$, it follows from the previous lemma that

$$\lim \| z_k x_i^* x_i z_k \| = \| R_\varphi(x_i^*x_i) \| .$$

Hence if $\| R_\varphi(x_j^*x_j) \| < 1$ for some j, then for large k, $\| z_k x_j^* x_j z_k \| < 1$ and so

$$z_{k+1}^2 - z_{k+1} x_j^* x_j z_{k+1} \geq c z_{k+1}^2$$

where $c = 1 - \| z_k x_j^* x_j z_k \| > 0$. Since

$$z_{k+1}^2 = z_{k+1} a z_{k+1} = \sum_{i=1}^{m} z_{k+1} x_i^* x_i z_{k+1} ,$$

this estimate implies that

$$c z_{k+1}^2 \leq \sum_{i \neq j} z_{k+1} x_i^* x_i z_{k+1} .$$

By using this we can reduce m by 1 as in [9] and so reach a contradiction. Thus one must have that $\| R_\varphi(x_i^*x_i) \| = 1$ for all i. As $R_\varphi(x_i^*x_i)$ is a positive matrix it follows that $\mathrm{Tr}\, R_\varphi(x_i^*x_i) \geq 1$ and that

$$m \leq \mathrm{Tr} \left\{ \sum_{i=1}^{m} R_{\varphi}(x_i^* x_i) \right\} = d.$$

The proof of (IV) is again routine if we use the fact that A^{α} is prime and has no minimal projections. So we omit it and refer to [9].

We shall not give a proof under condition (ii) except:

4.5. Proposition. Under condition (ii) in 4.2 (in particular there is an invariant pure state ω of A such that $\pi_{\omega|A^{\alpha}}$ is faithful), A^{α} is prime and has no minimal projections.

Proof. Since $\omega | A^{\alpha}$ is pure, it follows that A^{α} is prime.

Suppose that A^{α} has a minimal projection e. Then α is ergodic on eAe, and there is an invariant pure state φ of eAe since $\omega | A^{\alpha} e A^{\alpha} \neq 0$. Then π_{φ} is faithful and it follows from [10] that φ is a tracial state, which implies that $eAe = \mathbb{C}e$, i.e, e is minimal in A.

There is a non-zero $x \in A_1^{\alpha}(u)$ for some non-trivial irreducible unitary matrix representation u of G. Since A^{α} is prime, there is an $a \in A^{\alpha}$ such that $eaxx^* \neq 0$. Thus $\sum x_i^* a^* eax_i \neq 0$. Again there is a $b \in A^{\alpha}$ such that

$$\sum_i x_i^* a^* eax_i \cdot be \neq 0.$$

This shows that $eax_i be \neq 0$ for some i, which contradicts that $eax_i be \in \mathbb{C}e$ since $eaxbe \in A_1^{\alpha}(u)$.

Incidentally we give:

4.6. Proposition. Let A be a C*-algebra, G a compact group, and α a faithful continuous action of G on A. Let ω be a pure invariant state of A. Then the following conditions are equivalent:

(i) $\pi_{\omega|A^{\alpha}}$ is faithful.

(ii) $\pi_{\omega} \vee U$ is faithful, where U is the canonical unitary representation of G on H_ω and $\pi_\omega \times U$ is the corresponding representation of the crossed product $A \times_\alpha G$.

Proof. (ii) $\Rightarrow$ (i). Let for each $\gamma \in \hat{G}$

$$P_\gamma = \int_G d\, \overline{Tr(\gamma)}\, \lambda(g)\, dg \in M(A \times_\alpha G)$$

where $d = \dim(\gamma)$ and $Tr(\gamma)$ is the character of γ. Since $A \times_\alpha G \supset A^\alpha \otimes C^*(G) \supset A^\alpha \otimes P_\iota$ and $\pi_\omega \times U$ is faithful, $(\pi_\omega \times U)|A^\alpha \otimes P_\iota$ is faithful. That is, $\rho|U(P_\iota)H_\omega$ is faithful, where $\rho = \pi_\omega | A^\alpha$.

(i) $\Rightarrow$ (ii). Let $I = \ker(\pi_\omega \times U)$ and let x be a positive element of I. Let $b \in A_1^\alpha(\gamma)$, with $\gamma \in \hat{G}$, where γ is also regarded as a fixed unitary matrix representation in its class. Then since $P_\iota bxb^*P_\iota$ $= P_\iota \mathcal{E}(bxb^*)P_\iota$, where $\mathcal{E}(a) = \int_G \alpha_g(a)dg$, and since $\rho|U(P_\iota)H_\omega$ is faithful, it follows that $P_\iota bxb^*P_\iota = 0$. Thus $xb_i^*P_\iota = 0$ for all i. Let J be the closed ideal generated by $b_i^*P_\iota b_i$ with $b \in A_1^\alpha(\gamma)$, $i = 1,2,\ldots, \dim(\gamma)$, $\gamma \in \hat{G}$. Then one must have $IJ = 0$.

Since A^α is prime, and α is faithful, it follows that the spectrum of α is $\hat{G}$. Since $P_\gamma b^*P_\iota b = b^*P_\iota b \neq 0$ for non-zero $b \in A_1^\alpha(\gamma)$ and P_γ's are minimal central projections of $C^*(G)$, it follows that $J \cap A^\alpha \otimes P_\gamma \neq 0$ for each $\gamma \in \hat{G}$. Since A^α is prime and $\sum P_\gamma = 1$ (in the multiplier algebra), J must be essential. Thus $I = 0$.

5. Endomorphisms

Let (A, G, α) be a C*-dynamical system with G compact. We denote by $M(A)$ the multiplier algebra of A and by the same α the unique extension of α to an action on $M(A)$. We let $\mathcal{U} = \mathcal{U}(G)$ be the set of irreducible unitary matrix representations of G. In this section we assume that for any $u \in \mathcal{U}$ there is a $v \in M(A)^{\alpha}_{1}(u)$ such that $vv^* = 1$ and $v^*v = 1 \otimes 1 \in M(A) \otimes M_d$ where $d = \dim u$. Let $\gamma \in \hat{G}$, a class of equivalent irreducible unitary representations. We fix a $u^{\gamma} \in \mathcal{U}$ in class γ and in turn $v \in M(A)^{\alpha}_{1}(u^{\gamma})$ and define an endomorphism φ_{γ} of A^{α} by

$$\varphi_{\gamma}(a) = vav^* = \sum_{i=1}^{d} v_i a v_i^*, \quad a \in A^{\alpha}$$

where $v = (v_1, \ldots, v_d)$. Thus we have a family $\{ \varphi_{\gamma} : \gamma \in \hat{G} \}$ of endomorphisms of the fixed point algebra A^{α} (cf [6 ~ 8]).

One typical example of such a C*-dynamical system is given by

<u>5.1. Proposition</u>. Let λ be the right regular representation of G on $L^2(G)$. Let H be an infinite-dimensional separable Hilbert space and denote by $K(H)$ the compact operators on H. Denote by β the action of G on the C*-algebra $C = K(L^2(G)) \otimes K(H)$ defined by $\beta_g = \operatorname{Ad} \lambda(g) \otimes \iota$ where ι is the identity automorphism.

Then for any finite-dimensional unitary representation u of G, there is a $v \in M(C)^{\alpha}_{1}(u)$ such that $vv^* = 1$ and $v^*v = 1 \otimes 1 \in M(C) \otimes M_d$ with $d = \dim u$.

Proof. Let $w_1, \ldots, w_d$ be isometries in $M(K(H)) = B(H)$ such that

$$\sum w_i w_i^* = 1 .$$

Regarding u_{ij}, a matrix component of u, as a multiplication operator on $L^2(G)$, let

$$v_i = \sum_{j=1}^{d} u_{ji} \otimes w_j \ .$$

Then $v = (v_1, \ldots, v_d)$ satisfies the required conditions.

Let ω be an invariant state of A. Define a unitary representation U of G on H_ω by

$$U_g \, \pi_\omega(x)\Omega_\omega = \pi_\omega \circ \alpha_g(x)\Omega_\omega, \ x \in A.$$

Denote by P_γ be the spectral projection of U corresponding to $\gamma \in \hat{G}$, i.e., P_γ is in the center U_G'' such that the representation UP_γ of G is in class γ. Note that $P_\gamma \neq 0$ for any $\gamma \in \hat{G}$.

<u>5.2. Lemma</u>. Let ω be an invariant state of A and let U, P_γ etc. be as above. Let $\pi_\omega' = \pi_\omega | A^\alpha$ and let $\rho = \pi_{\omega|A^\alpha} = \pi_\omega' \,|\, P_\iota H_\omega$. Then for each $\gamma \in \hat{G}$ it follows that $\pi_\omega' \,|\, P_\gamma H_\omega$ is unitarily equivalent to the direct sum of d copies of $\rho \circ \varphi_{\bar{\gamma}}$ where $d = \dim \gamma$ and $\bar{\gamma}$ is the conjugate class of γ.

Proof [7], [3]. Let $\gamma \in \hat{G}$ and let $v \in M(A)_1^\alpha(u)$ be the element defining φ_γ. It follows that $P_{\bar{\gamma}} H_\omega$ is the direct sum of $[\pi_\omega(v^*) H_\iota]$, $i = 1, \ldots, d$, each of which is left invariant under $\pi_\omega(A^\alpha)$, where $H_\iota = [\pi_\omega(A^\alpha)\Omega_\omega] = P_\iota H_\omega$. The restriction V_i of $\sqrt{d}\, \pi_\omega(v_i^*)$ to H_ι is an isometry onto $[\pi_\omega(v_i^*) H_\iota]$ and satisfies that

$$V_i \, \rho \circ \varphi_\gamma(a) = \pi_\omega'(a) V_i, \quad a \in A^\alpha.$$

This completes the proof.

<u>5.3. Proposition</u>. Let (A, G, α) be as above; in particular there is a family $\{ \varphi_\gamma : \gamma \in \hat{G} \}$ of endomorphisms of A^α. Let ω be an extreme invariant state of A (and so $\omega | A^\alpha$ is pure). Then the following conditions are equivalent:

(i) ω is pure.

(ii) For each $\gamma \in \hat{G} \setminus \{\iota\}$, $\rho \circ \varphi_\gamma$ is irreducible and disjoint from ρ.

(iii) For each $\gamma \in \hat{G} \setminus \{\iota\}$, $\rho \circ \varphi_\gamma$ is disjoint from ρ.

Proof of (i) $\Rightarrow$ (ii). Let $\gamma \in \hat{G}$. In the proof of 5.2 it follows that $\pi'_\omega \mid [\pi_\omega(v_i^*)H_\iota]$ is irreducible since $U_G''P_\gamma \cong M_d$ and $U_G'P_\gamma = \pi'_\omega(A^\alpha)''P_\gamma$. Since $\pi'_\omega \mid [\pi_\omega(v_i^*)H_\iota]$ is unitarily equivalent to $\rho \circ \varphi_\gamma$ $(= \pi'_\omega \circ \varphi_\gamma \mid H_\iota)$, it follows that $\rho \circ \varphi_\gamma$ is irreducible. Since $\pi'_\omega \mid P_{\bar{\gamma}} H_\omega$ is disjoint from $\pi'_\omega \mid P_\iota H_\omega$, $\rho \circ \varphi_\gamma$ is disjoint from ρ.

Proof of (ii) $\Rightarrow$ (iii). This is trivial.

Proof of (iii) $\Rightarrow$ (i). Using the notation in the proof of 5.2, one has that $P_\iota \in \pi_\omega(A^\alpha)''$, $P_\iota \pi_\omega(A)'' P_\iota = B(P_\iota H_\omega)$ and the central support of P_ι is 1. Hence $\pi_\omega(A)' = \mathbb{C}1$ and so ω is pure.

By an analogy from the case of automorphisms we define

5.4. Definition. Let φ be an endomorphism of a C*-algebra B. One calls φ to be properly outer if for any non-zero hereditary C*-subalgebra D of B and any a of B it follows that

$$\inf \{ \|xa\varphi(x)\| \ : \ x \in T \cap D \} = 0.$$

5.5. Lemma. Let $\{\varphi_n : n = 1, 2, \ldots\}$ be a sequence of endomorphisms of a separable prime C*-algebra B. Suppose that all φ_n are properly outer. Then there exists a faithful irreducible representation π of B such that $\pi \circ \varphi_n$ is disjoint from π for any n.

Proof. Let $\{I_n\}$ be a decreasing sequence of non-zero ideals of B such that for any non-zero ideal J of B there is an n such that $J \supset I_n$. Let $\{a_n\}$ be a dense sequence in B. Enumerate $\{(a_k, \varphi_m) : k, m = 1, 2, \ldots\}$ and let $\{(a_n, \psi_n)\}$ be the resulting sequence. As in [11], one constructs a decreasing sequence $\{e_n\}$ such that $e_n \in T \cap I_n$, $e_n e_{n+1} = e_{n+1}$, and

$$\| e_n a_n \psi_n(e_n) \| < 1/n.$$

Let f be a pure state of B such that $f(e_n)=1$ for all n. Then π_f is the desired representation.

5.6. Lemma. Let φ be an endomorphism of B. If there is a faithful irreducible representation π of B such that $\pi \circ \varphi$ is disjoint from π, then φ is properly outer.

Proof. Trivial.

5.7. Proposition. Let (A,G,α) be as above, and suppose that A is separable. Then the following conditions are equivalent:

(i) There is a pure invariant state ω of A such that $\pi_{\omega|A^\alpha}$ is faithful.

(ii) A^α is prime and φ_γ is properly outer for any $\gamma \in \hat{G} \setminus \{\iota\}$.

Proof. (i) $\Rightarrow$ (ii) follows from 5.3 and 5.6. (ii) $\Rightarrow$ (i) follows from 5.3 and 5.5.

5.8. Lemma. Let (A, G, α) be as above. Suppose that there is a faithful irreducible representation π of A such that $\pi \mid A^\alpha$ is irreducible. Then for any $\gamma \in \hat{G}$ and for any $x, y \in T \cap A$, it follows that

$$\sup \{ \| x \varphi_\gamma(a) y \| : a \in A, \| a \| \leq 1 \} \geq d^{-4}.$$

Proof. Assume that φ_γ is defined by $v \in M(A)_1^\alpha(u^\gamma)$, i.e., $\varphi_\gamma(x) = vxv^*$. First we claim that $\pi \circ \varphi_\gamma(A^\alpha)'$ is the C*-algebra $\mathcal{M}$ generated by $\pi(v_i v_j^*)$, i,j = 1,...,d, which is isomorphic to M_d.

It is clear that $\mathcal{M} \cong M_d$ since $\pi(v_i v_j^*)$'s are matrix units. Define an endomorphism $\tilde{\varphi}$ of $B(H_\pi)$ by

$$\tilde{\varphi}(Q) = \sum_{i=1}^{d} \pi(v_i) Q \pi(v_i^*) .$$

Then it follows that the image of $\tilde{\varphi}$ is $\mathcal{M}'$. Since $\pi | A^\alpha$ is irreducible, it follows that $\pi\circ\varphi(A^\alpha)''$ contains any element of the form $\tilde{\varphi}(Q)$, $Q \in B(H_\pi)$.

Let $x, y \in T \cap A^\alpha$ and let

$$\delta = \sup\{\|x\varphi(a)y\| : a \in A_1^\alpha\}$$

where $A_1^\alpha = \{a \in A^\alpha : \|a\| \leq 1\}$. Then

$$\delta = \sup\{\|\pi(x)\tilde{\varphi}(Q)\pi(y)\| : Q \in B(H_\pi)_1\}.$$

Take a partial isometry u for Q where $p = uu^*$ and $q = u^*u$ are one-dimensional projections. Then it follows that

$$\delta \geq \|\tilde{\varphi}(p)\pi(x)\tilde{\varphi}(u)\pi(y)\tilde{\varphi}(q)\|.$$

Since $\pi(x) = \sum \tilde{\varphi}(\pi(v_i^* x v_j))\pi(v_i v_j^*)$, one obtains that

$$\delta \geq \sum_{i,k,j} \tilde{\varphi}(p\pi(v_i^* x v_k)p)\tilde{\varphi}(u)\tilde{\varphi}(q\pi(v_k^* y v_j))\pi(v_i v_j^*)$$

Define states f_p, f_q of $B(H_\pi)$ by

$$f_p(Q)p = pQp, \quad f_q(Q)q = qQq, \quad Q \in B(H_\pi).$$

Then defining $a_{ij} = f_p(\pi(v_i^* x v_j))$ and $b_{ij} = f_q(\pi(v_i^* y v_j))$, one obtains

$$\delta \geq \|\sum_{i,j}\sum_k a_{ik}b_{kj}\pi(v_i v_j^*)\| = \|A\cdot B\|,$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are $d\times d$ matrices. The above inequality is true for those A and B defined by vector states f_p, f_q of $B(H_\pi)$ and hence for A and B defined by any normal states. It then follows by taking a weak* limit that for any states f and g of A

$$\delta \geq \|(f(v_i^* x v_j))\cdot(g(v_i^* y v_j))\|.$$

Note that $\|x^{1/2} v_i\| \geq d^{-1/2}$ since

$$\int \alpha_g(x^{1/2} v_i v_i^* x^{1/2})dg = d^{-1}x.$$

This implies that $\|\sum v_i^* x v_i\| \geq 1/d$ and hence there are invariant states f and g of A such that

$$f(\sum v_i^* x v_i) = \|\sum v_i^* x v_i\| \geq 1/d$$
$$g(\sum v_i^* y v_i) = \|\sum v_i^* y v_i\| \geq 1/d.$$

Since $f(v_j^* x v_j) = 1/d\, f(\sum v_i^* x v_i)$ and $f(v_i^* x v_j) = 0$ for $i \neq j$, it follows that

$$\delta \geq d^{-2} \| \sum v_i^* x v_i \| \, \| \sum v_i^* y v_i \| \geq d^{-4} .$$

<u>5.9. Proposition.</u> Let A be a separable C^*-algebra, G a compact group with $G \neq \{e\}$, and α a faithful continuous action of G on A. Suppose that for each $\gamma \in \hat{G}$ with fixed unitary matrix representation u^γ in class γ, there is a $v \in M(A)_1^\alpha(u^\gamma)$ such that $vv^* = 1$, $v^*v = 1 \otimes 1_d$ with $d = \dim \gamma$, defining an endomorphism φ_γ of A^α by

$$\varphi_\gamma(x) = vxv^* = \sum_{i=1}^{d} v_i x v_i^* \quad , \quad x \in A^\alpha .$$

Also suppose that there is a faithful irreducible representation π of A such that $\pi \mid A^\alpha$ is irreducible. Then φ_γ is properly outer for any $\gamma \in \hat{G} \setminus \{\iota\}$.

Proof. Let $\gamma \in \hat{G} \setminus \{\iota\}$ and suppose that φ_γ is not properly outer. Thus there are a non-zero hereditary C^*-subalgebra D of A and $a \in A^\alpha$ such that $\| x a \varphi_\gamma(x) \| \geq 1$ for any $x \in T \cap D$.

Let $\xi \in [\pi(D)H_\pi]$ be a unit vector and define a state ω of A by $\omega(x) = \langle \pi(x)\xi , \xi \rangle$. Since $\omega | A^\alpha$ is pure, there is a decreasing sequence $\{z_k\}$ in $T \cap A$ such that $z_k z_{k+1} = z_{k+1}$, and the limit p of z_k in $(A^\alpha)^{**}$ is the support (minimal) projection of $\omega | A^\alpha$. Since $\| z_k a v z_k \| \geq 1$ for any k where $\varphi_\gamma = \mathrm{Ad}\, v$, one obtains by 4.4 that

$$\sup \{ \| R_{\omega \circ \alpha_g}(av) \| \ : g \in G \} \geq 1 .$$

But as $R_{\omega \circ \alpha_g}(av) = R_\omega(av)u(g)$, it follows that $\| R_\omega(av) \| \geq 1$.

Suppose that $d = \dim \gamma = 1$. (In this case we can use a method as in [11].) Then v is a unitary in $M(A)$ and

$$| \langle \pi(av) \xi , \xi \rangle | \geq 1$$

for any unit vector $\xi \in [\pi(D)H_\pi]$.

Let E be the open projection in A^{**} corresponding to D. Then the above condition implies that there is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that

$$\lambda\, \pi^{**}(E\, a\, V\, E) + \bar{\lambda}\, \pi^{**}(E\, V^*\, a^*\, E) \geq 2\, \pi^{**}(E).$$

Let $x \in D$. Then, since π is faithful, it follows that

$$\lambda\, x\, a\, v\, x^* + \bar{\lambda}\, x\, v^*\, a^*\, x^* \geq 2\, x\, x^* .$$

By applying α_g and integrating it over G, one reaches the contradiction that

$$0 \geq 2\, x\, x^* \quad , \quad x \in D .$$

Suppose that $d \geq 2$. Then by 3.1 and 5.8, there is a faithful irreducible representation ρ of A^α such that $\rho \,|\, \varphi_\gamma(A^\alpha)$ is irreducible.

Let ξ be a unit vector in $[\rho(D)H_\rho]$ and define a state ω of A^α by $\omega(x) = \langle \rho(x)\xi , \xi \rangle$. Let $\{z_k\}$ be a sequence in $T \cap D$ as before. Then $\| z_k a\, \varphi_\gamma(z_\ell) \| \geq 1$ for all k, ℓ and hence

$$\omega(a\, \varphi_\gamma(z_\ell) a^*) \geq 1.$$

This implies that $(\rho \circ \varphi_\gamma)^{**}(p) \neq 0$ where p is the support projection of ω, i.e., the representation $\rho \circ \varphi_\gamma$ of A^α contains ρ as ρ is equivalent to the GNS representation associated with ω. But since $\rho \circ \varphi_\gamma$ is irreducible, this means that $\rho \circ \varphi_\gamma$ is equivalent to ρ and that there is a unitary U on H_ρ such that $\rho \circ \varphi_\gamma(a) = U \rho(a) U^*$, $a \in A$. Hence $\rho \circ \varphi_\gamma^2(a) = U \rho \circ \varphi_\gamma(a) U^* = U^2 \rho(a) U^{*2}$, $a \in A$. Thus $\rho \,|\, \varphi_\gamma^2(A^\alpha)$ is irreducible.

Note that φ_γ^2 is an endomorphism of A^α associated with $u^\gamma \otimes u^\gamma$ which is not irreducible. Explicitely let $p \in M(A^\alpha)$ be the projection corresponding to the symmetric part:

$$p = \sum_{i<j} 2^{-1} (v_i v_j + v_j v_i)(v_i v_j + v_j v_i)^* + \sum_k v_k v_k v_k^* v_k^* .$$

Then $\alpha_g(p) = p$ and $0 \lneqq p \lneqq 1$ and one has that $p\, \varphi_\gamma^2(A^\alpha)(1-p) = \{0\}$.

Since $\rho(p) \neq 0$ and $\rho(1-p) \neq 0$, this implies that $\rho \mid \varphi_{\gamma}^{2}(A^{\alpha})$ cannot be irreducible, i.e., a contradiction.

5.10. Remark. To complete the proof of (i) $\Rightarrow$ (ii) in Theorem 1.1, we must first replace (A, G, α) by $(A \otimes K(L^2(G)) \otimes K(H), G, \alpha \otimes \mathrm{Ad}\,\lambda \otimes \iota)$ (see 5.1).

Checking that (i) is still satisfied for this new system, we apply 5.9 and 5.7 and then have to go back to the original system.

6. Infinite tensor product type actions

Let G be a compact group with countable basis. For each $\gamma \in \hat{G}$ we fix a unitary matrix representation u^{γ} in class γ. Let $\{\xi_n\}$ be a sequence of representations $1 \oplus u^{\gamma}$, $\gamma \in \hat{G}$ such that each $1 \oplus u^{\gamma}$ appears infinitely often in $\{\xi_n\}$ where 1 is the trivial one-dimensional representation. Let $d_n = \dim \xi_n$ and let β be the infinite tensor product action $\bigotimes_{n=1}^{\infty} \mathrm{Ad}\, \xi_n$ of G on the UHF algebra $C = \bigotimes_{n=1}^{\infty} M_{d_n}$.

Let p_n be the one-dimensional projection of M_{d_n} that supports the trivial representation 1. Regarding p_n as a projection of C, let ω be the pure state of C whose support projection is the limit of $p_1 \ldots p_n$ in C**.

Let q_n be the one-dimensional projection of M_{d_n} such that every matrix entry of ξ_n is d_n^{-1} (in the matrix factor M_{d_n} where ξ_n is represented as it is). Regarding q_n as a projection of C let φ be the pure state of C whose support projection is the limit of $q_1 \ldots q_n$ in C**.

6.1. Proposition. Let (C, G, β) be as above, and let ω and φ be the pure states as defined above. Then (a) ω is a pure invariant state such that $\pi_{\omega|A^{\alpha}}$ is faithful and (b) φ is a pure 'anti-invariant' state in the sense that $\pi_{\varphi} | A^{\alpha}$ is irreducible.

We shall prove here part (a). Let I be the kernel of $\pi_{\omega|A^{\alpha}}$ and suppose that $I \neq \{0\}$. Then we must have that $I_n \equiv I \cap C_n^{\beta} \neq \{0\}$ for some n where $C_n = \bigotimes_{k=1}^{n} M_{d_k}$. Since C_n^{β} is the direct sum of finite type I factors, each corresponding to each central direct summand of $\bigotimes_{k=1}^{n} \xi_k$, and I_n is an ideal of C, there is a minimal central projection e of C_n^{β} with $e \in I$. Let $\gamma \in \hat{G}$ be the class containg $(\bigotimes_{k=1}^{n} \xi_k)e$. Then there is an $x \in (C_n)_1^{\beta}(u^{\gamma})$ such that $x^*x = p_1 \ldots p_n \otimes 1$ and

$x x^* \leq e$. On the other hand there is an $m > n$ such that $\xi_m \cong 1 \ni \overline{u^\gamma}$ and hence there is a $v \in (M_{d_m})_1^\beta(\overline{u^\gamma})$ such that $v^* v = P_m \otimes 1$ and $v v^* = 1 - P_m$. Then it follows that $a = \sum x_i v_i \in C^\beta$ and that

$$\begin{aligned} \omega(a^* e a) &= \sum \omega(x^*_i e x_j)\, \omega(v^*_i v_j) \\ &= \sum \omega(x^*_i e x_i) \\ &= \dim(\gamma) \end{aligned}$$

This contradicts the assumption that $e \in I$. Hence $\pi_{\omega|A^\alpha}$ must be faithful.

Part (b) can be proved similarly. See [9], [2].

7. Central sequences

We shall give the proof of (i) $\Rightarrow$ (v) in Theorem 1.1. (The result here relies on Lemma 1.1 in [1].)

Suppose (i) and let π be a representation as in (i). Let $\gamma \in \hat{G}$ and let u be a unitary matrix representation in class γ. Define a representation $\tilde{\pi}$ of A by

$$\tilde{\pi} = \int_G^{\oplus} \pi \circ \alpha_g \, dg$$

as before. Then $\tilde{\pi}(A)'' \ni \langle u\xi, \xi \rangle$ for any unit vector $\xi \in \mathbb{C}^d$ with d = dim u, where $\langle u\xi, \xi \rangle$ is regarded as the multiplication operator by $\langle u(t)\xi, \xi \rangle$. Hence there exists a central sequence $\{y^{(n)}\}$ in A such that $\|y^{(n)}\| \leq 1$ and $\tilde{\pi}(y^{(n)})$ converges to $\langle u\xi, \xi \rangle$ in the strong* topology [1].

Define

$$y_{ij}^{(n)} = d \int_G u_{ij}(s) \alpha_{s^{-1}}(y^{(n)}) \, ds,$$

and let $\underset{\sim}{y}^{(n)} = (y_{ij}^{(n)})$. Then it follows that $\alpha_s(\underset{\sim}{y}^{(n)}) \equiv (\alpha_s(y_{ij}^{(n)})) = \underset{\sim}{y}^{(n)}u(s)$. By computation one obtains that for $\eta, \zeta \in \mathbb{C}^d$,

$$\tilde{\pi}(\eta^* \underset{\sim}{y}^{(n)} \zeta) \rightarrow \langle \xi, \eta \rangle \langle u\zeta, \xi \rangle \quad \text{strongly*}$$

which implies that $\tilde{\pi}(\underset{\sim}{y}^{(n)})$ converges to $\xi\xi^* u$ on $L^2(G) \otimes H_\pi \otimes \mathbb{C}^d$ in the strong* topology, where u is the multiplication operator by the matrix valued function u(t). Hence for $\Phi \in H_\pi$ and $\eta \in \mathbb{C}^d$, it follows that

$$\| \pi \circ \alpha_s(\underset{\sim}{y}^{(n)})(\Phi \otimes u(s)^* \eta) - \xi\xi^* u(s)(\Phi \otimes u(s)^* \eta) \|$$
$$= \| \pi(\underset{\sim}{y}^{(n)}) \Phi \otimes \eta - \xi\xi^* \Phi \otimes \eta \|$$

and that

$$\|\tilde{\pi}(\underset{\sim}{y}^{(n)})(\Phi\otimes u^*\eta) - \xi\xi^*u(\Phi\otimes u^*\eta)\|$$
$$= \|\pi(\underset{\sim}{y}^{(n)})(\Phi\otimes\eta) - \xi\xi^*(\Phi\otimes\eta)\|$$

where $\Phi\otimes u^*\eta$ is the vector of $L^2(G)\otimes H_\pi\otimes C^d$ defined by $t\mapsto \Phi\otimes u(t)^*\eta$. This implies that $\pi(\underset{\sim}{y}^{(n)})$ converges to $\xi\xi^*$ in the strong* topology.

Let $a^{(n)} = \underset{\sim}{y}^{(n)}\underset{\sim}{y}^{(n)*}$. Then $a^{(n)}$ is a positive element of $A^\alpha\otimes M_d$. Define a function f on $\mathbb{R}$ by

$$f(t) = \begin{cases} t^{-1/2} & t \geq 1 \\ t & t < 1 \end{cases}$$

and let $z^{(n)} = f(a^{(n)})\underset{\sim}{y}^{(n)}$. Then $\alpha_s(z^{(n)}) = z^{(n)}u(s)$ and $\|z^{(n)}\| = 1$. Since

$$\pi(a^{(n)}) \to \xi\xi^* \quad \text{strongly}$$

and since $\|\xi\xi^*\| = 1$, it also follows (see e.g., [18]) that

$$\pi(f(a^{(n)})) \to \xi\xi^*.$$

Thus $\pi(z^{(n)}) \to \xi\xi^*$ strongly*. Hence $\xi^*z^{(n)} \in A_1^\alpha(u)$, $\|\xi^*z^{(n)}\xi\| \leq \|\xi\|\,\|z^{(n)}\|\,\|\xi\| = 1$, $\{\xi^*z^{(n)}\xi\}$ is a central sequence, and $\pi(\xi^*z^{(n)}\xi) \to 1$, which implies that

$$\liminf \|\xi^*z^{(n)}\xi\cdot a\| = \|a\|$$

for any $a\in A$.

Incidentally we remark the following.

<u>7.1. Proposition</u>. Let A be a simple unital C*-algebra and $\{z_n\}$ a central sequence such that $\|z_n\| = 1$. Then for any $a\in A$,

$$\lim \|z_n a\| = \|a\|.$$

We omit the (easy) proof of this result (cf. [16]).

8. Asymptotic abelianess

In this section we show how the asymptotic abelianess condition can be used to produce an 'anti-invariant' pure state, i.e., a pure state whose associated GNS representation generates a type I orbit as in the following (cf. [4]):

8.1. Theorem. Let A be a separable simple unital C^*-algebra, G a locally compact group with countable basis, and α a faithful continuous action of G on A. Suppose that there is an automorphism σ of A such that $\sigma \circ \alpha_t = \alpha_t \circ \sigma$, $t \in G$ and $\| [x, \sigma^n(y)] \| \to 0$ as $n \to \infty$ for all $x, y \in A$. Then there is an irreducible representation π of A such that for the representation of A defined by

$$\tilde{\pi} = \int_G^{\oplus} \pi \circ \alpha_t \, dt$$

on $L^2(G) \otimes H_\pi$, the center of $\tilde{\pi}(A)''$ is $L^\infty(G) \otimes \mathbb{C}1$.

Proof. For $f \in L^1(G)$ we define a linear map α_f on A by

$$\alpha_f(x) = \int_G f(t)\, \alpha_t(x)\, d\mu(t), \quad x \in A$$

where $d\mu$ is a right Haar measure on G. Since $\|\alpha_f\| \le \|f\|_1$, and $\alpha_f \circ \alpha_g = \alpha_{f*g}$ with $f*g(s) = \int f(t) g(t^{-1}s)\, d\mu(t)$, the map $f \mapsto \alpha_f$ is a continuous homomorphism of $L^1(G)$ into the bounded maps on A. We first claim that this map is an injection, i.e.,

$$I = \{ f \in L^1(G) : \alpha_f = 0 \}$$

is the zero ideal of $L^1(G)$.

Since I is a closed ideal, $I \cap C_0(G)$ is dense in I, where $C_0(G)$ is the continuous functions on G vanishing at infinity. Let $f \in I \cap C_0(G)$. Then for any $x, y \in A \setminus \{0\}$, it follows that

$$\int f(t)\, \alpha_t(x\, \sigma^n(y))\, d\mu(t) = 0$$

for $n = 1, 2, \ldots$.

For any $\varepsilon > 0$, let K be a compact subset of G such that

$$\int_{G \setminus K} |f(t)|\, d\mu(t) < \varepsilon .$$

Let U be an open neighbourhood of e of G such that

$$\| \alpha_t(x) - x \| < \varepsilon / \mu(K) \|y\| M, \qquad \| \alpha_t(y) - y \| < \varepsilon / \mu(K) \|x\| M$$

for all $t \in U$, where $M = \sup \{ |f(t)| : t \in K \}$, and

$$|f(st) - f(s)| < \varepsilon / \|x\| \cdot \|y\| \cdot \mu(K)$$

for any $s \in K$ and $t \in U$. There are $t_1, \ldots, t_n$ in G such that

$$\bigcup_{i=1}^{n} t_i U \supset K.$$

Define for $i = 1, \ldots, n$,

$$A_i = (t_i U \cap K) \setminus \bigcup_{j=0}^{i-1} A_j$$

with $A_0 = \phi$. For $t \in U$

$$\| f(t_i t)\, \alpha_{t_i t}(x\, \sigma^n(y)) - f(t_i)\, \alpha_{t_i}(x \sigma^n(y)) \| \leq 3\varepsilon / \mu(K).$$

Let $\lambda_i = \mu(A_i)$. Then

$$\begin{aligned} &\| \int f(t)\, \alpha_t (x\, \sigma^n(y))\, d\mu(t) - \sum_{i=1}^{n} \lambda_i f(t_i)\, \alpha_{t_i}(x\, \sigma^n(y)) \| \\ \leq\ &\sum_i \int_{A_i} \| f(t)\, \alpha_t(x\, \sigma^n(y)) - f(t_i)\, \alpha_{t_i}(x\, \sigma^n(y)) \|\, d\mu(t) \\ \leq\ &\sum_i (3\varepsilon / \mu(K))\, \mu(A_i) = 3\varepsilon . \end{aligned}$$

Therefore one obtains that

$$\| \sum_{i=1}^{n} \lambda_i f(t_i)\, \alpha_{t_i}(x\, \sigma^n(y)) \| \leq 3\varepsilon$$

for all $n = 1, 2, \ldots$. As $n \to \infty$ one obtains that

$$\| \sum \lambda_i f(t_i)\, \alpha_{t_i}(x) \otimes \alpha_{t_i}(y) \|_{\gamma_2} \leq 3\varepsilon$$

where γ_2 is a C*-norm on $A \otimes A$ (cf. [16]).

On the other hand, for the same reasoning as above, one has that

$$\left\| \int f(t)\, \alpha_t(x) \otimes \alpha_t(y)\, d\mu(t) - \sum \lambda_i f(t_i)\, \alpha_{t_i}(x) \otimes \alpha_{t_i}(y) \right\|_{\gamma_2} \leq 3\,\varepsilon.$$

Hence it follows that

$$\left\| \int f(t)\, \alpha_t(x) \otimes \alpha_t(y)\, d\mu(t) \right\| \leq 6\,\varepsilon.$$

Since ε is arbitrary, this implies that

$$\int f(t)\, \alpha_t(x) \otimes \alpha_t(y)\, d\mu(t) = 0$$

In a similar way one can show that for $f \in I$,

$$\int f(t)\, \alpha_t(x_1) \otimes \cdots \otimes \alpha_t(x_n)\, d\mu(t) = 0$$

for any $x_1, \ldots, x_n \in A$. In other words, one has that

$$\int f(t)\, \varphi_1(\alpha_t(x_1)) \cdots \varphi_n(\alpha_t(x_n))\, d\mu(t) = 0$$

for any $x_1, \ldots, x_n \in A$ and $\varphi_1, \ldots, \varphi_n \in A^*$. Since the set $\mathcal{F}$ of functions $t \mapsto \varphi(\alpha_t(x))$ on G with $x \in A$ and $\varphi \in A^*$ separates the points of G, and is closed under the complex conjugation, it generates $L^\infty(G)$ as a $\sigma(L^\infty(G), L^1(G))$-closed algebra, as is seen below.

Let $\mathcal{F}_1$ be the uniformly closed algebra generated by $\mathcal{F}$. Then for a continuous function h on $\mathbb{R}$ with compact support $h \circ \varphi$ belongs to $\mathcal{F}_1$ for all real $\varphi \in \mathcal{F}_1$, since φ is bounded and any h can be uniformly approximated by polynomials on a bounded interval. Let $g \in L^\infty(K)$ be a real valued function with K compact. Then there exists a sequence $\{\varphi_n\}$ in $\mathcal{F}_1$ such that $\varphi_n | K \to g$ in $\sigma(L^\infty(K), L^1(K))$, $\varphi_n^* = \varphi_n$, and $\| \varphi_n | K \| \leq \| g \|$. Then by replacing φ_n by $h \circ \varphi_n$ if necessary we can assume that $\{ \|\varphi_n\| \}$ is bounded. Thus there exists a φ in the σ-weak closure of $\mathcal{F}_1$ such that $\varphi | K = g$. Since K is arbitrary, this shows that the σ-weak closure of $\mathcal{F}_1$ is equal to $L^\infty(G)$.

Hence for any $\varphi \in L^{\infty}(G)$,

$$\int f(t)\varphi(t)\,d\mu(t) = 0,$$

which implies that $f = 0$. This concludes the proof of $I = \{0\}$.

Let $\{f_n\}$ be a dense sequence in $\{f \in L^1(G) : \|f\| = 1\}$. For each n, there is an $x_n \in A$ such that $\|x_n\| = 1$ and

$$\sup \operatorname{Spec}(\alpha_{f_n}(x_n) + \alpha_{f_n}(x_n)^*) > \|\alpha_{f_n}\|/2.$$

Let $\{c_n\}$ be a sequence which consistes of infinitely many copies of $\alpha_{f_n}(x_n)$ with $n = 1, 2, \ldots$. Let $\{a_n\}$ be a dense sequence in $\{a \in A : \|a\| = 1\}$. We choose k_n and b_n, $e_n \in T$ with $e_0 \in T$ arbitrarily fixed, in the following way:

$$\|[a_\ell, \sigma^{k_n}(x_m)]\| < 1/n, \quad \ell, m = 1, 2, \ldots, n,$$

$$b_n \in H(e_{n-1}) \cap T,$$

$$\sup \operatorname{Spec}\{b_n \sigma^{k_n}(c_n + c_n^*)b\} > (1 - 1/2n)\|c_n + c_n^*\|,$$

$$e_n = g_n(b_n \sigma^{k_n}(c_n + c_n^*)b_n),$$

where $g_n(t)$ is defined by

$$g_n(t) = \begin{cases} 1 & t \geq (1 - 1/2n)\|c_n + c_n^*\| \\ 0 & t \leq (1 - 1/n)\|c_n + c_n^*\| \end{cases}$$

and by linearity elsewhere. Then $e_n \in H(e_{n-1}) \cap T$. Let φ be a pure state of A such that $\varphi(e_n) = 1$ for all n. Then

$$\operatorname{Re}\varphi(\sigma^{k_n}(c_n)) \geq 2^{-1}(1 - 1/2n)\|c_n + c_n^*\|.$$

For each m let $\{\ell_n\}$ be a subsequence such that $c_{\ell_n} = \alpha_{f_m}(x_m)$. Let g_m be a weak* limit point of $\tilde{\pi}_\varphi(\sigma^{k_{\ell_n}}(x_m))$. Since $\{\sigma^{k_{\ell_n}}(x_m)\}_n$ is a central sequence, g_m is a central element of $\tilde{\pi}_\varphi(A)''$ with norm less than or equal to 1 and can be regarded as a function on G. It follows that

$$\mathrm{Re}\int f_m(t)g_m(t)\,d\mu(t) \geq 4^{-1}\,\|\alpha_{f_m}\|.$$

Let $Z = \tilde{\pi}_\varphi(A)'' \cap \tilde{\pi}_\varphi(A)' \subset L^\infty(G) \otimes \mathbb{C}1$. Assume that $Z \subsetneq L^\infty(G) \otimes \mathbb{C}1$ and let $f \in L^1(G)$ be such that $\|f\| = 1$ and

$$\int f(t)g(t)\,d\mu(t) = 0, \quad g \in Z.$$

Then there is a subsequence $\{m_n\}$ such that $\|f_{m_n} - f\|_1 \to 0$. It follows that

$$4^{-1}\,\|\alpha_{f_m}\| \leq \mathrm{Re}\int f_m(t)g_m(t)\,d\mu \leq \|f_m - f\|_1\,\|g_m\|_\infty \leq \|f_m - f\|_1$$

and

$$\|\alpha_f\| \leq \|\alpha_f - \alpha_{f_m}\| + \|\alpha_{f_m}\| \leq \|f - f_m\|_1 + 4\,\|f - f_m\|_1$$
$$= 5\,\|f - f_m\|_1 .$$

Thus one obtains that $\alpha_f = 0$, which implies the contradiction that $f = 0$. Hence $Z = L^\infty(G) \otimes \mathbb{C}1$.

9. Other type I orbits

In this section we shall prove the following result based on 1.1.

9.1. Theorem. Let A be a separable C*-algebra, G a compact group, and α a faithful continuous action of G on A. Let H be a closed subgroup of G. Then the following conditions are equivalent:

(i) There exists a faithful irreducible representation π of A such that $\pi | A^{\alpha}$ is irreducible.

(ii) A^H is prime and there exists an H-invariant pure state φ of A such that $\pi_{\varphi|A^H}$ is faithful and $\tilde{\pi}_{\varphi}(A)''$ is of type I with center $L^{\infty}(H \setminus G) \otimes \mathbb{C}1$ where A^H is the fixed point algebra of $\alpha | H$ and

$$\tilde{\pi}_{\varphi} = \int_G^{\oplus} \pi_{\varphi} \circ \alpha_s \, ds .$$

9.2. Lemma. For an irreducible unitary matrix representation u of G let

$$P_u^H = P_u = \int_H u(h)dh.$$

Then A^H is the closed linear span of the set of $(yP_u)_i$, $i = 1,\dots,\dim u$, with $y \in A_1^{\alpha}(u)$ and u all of those representations of G.

Proof. Note that P_u is a projection and that A is the closed linear span of y_i , $i = 1,\dots, \dim u$ with $y \in A_1^{\alpha}(u)$ and $u \in \mathcal{U}$, where $\mathcal{U}$ is the set of all irreducible unitary matrix representations of G.

Thus A^H is the closed linear span of

$$\int_H \alpha_h(y_i)dh = \int (yu(h))_i dh = (yP_u)_i ,$$

$i = 1, \dots, \dim u$, with $y \in A_1^{\alpha}(u)$ and $u \in \mathcal{U}$.

Proof of (i) $\Rightarrow$ (ii) of 9.1. Since $A^H \supset A^G \equiv A^{\alpha}$, $\pi | A^H$ is irreducible if $\pi | A^G$ is irreducible. Thus A^H is prime.

By using a representation π as in (i), we define a representation Φ of the crossed product $A \times_{\beta} H$, with $\beta = \alpha | H$, on $L^2(H, H_{\pi})$ by

$$(\Phi(a)\xi)(t) = \pi \circ \alpha_t(a)\xi(t), \quad a \in A$$

$$(\Phi(\lambda(s))\xi)(t) = \xi(ts), \quad s \in H$$

for $\xi \in L^2(H, H_\pi)$, where λ is the canonical unitary representation of H in the multiplier algebra $M(A \times_\beta H)$ and the unique extension of Φ to $M(A \times_\beta H)$ is denoted by the same symbol Φ.

First we assert that Φ is a faithful irreducible representation of $A \times_\beta H$. Since π is faithful, Φ is faithful. Since $\pi | A^H$ is irreducible, the center of $\Phi(A)''$ is $L^\infty(H) \otimes \mathbb{C}1$ on $L^2(H) \otimes H_\pi$ $= L^2(H, H_\pi)$. Thus Φ is irreducible.

Let $\mathcal{U}$ be the set of all irreducible unitary matrix representations of G as before. Let $u \in \mathcal{U}$ be such that $u_{11}(h) = 1$ for $h \in H$. Then there is a central sequence $\{x(u, k)\}$ (central in A) in $\{x_1 : x \in A_1^\alpha(u)\}$ such that $\| x(u, k) \| = 1$ and

$$\limsup_k \| a\, x(u, k) \| \geq \delta_{[u]} \| a \|$$

for any $a \in A$, where $\delta_{[u]}$ is a positive constant depending only on the class of u (see (iv) in Theorem 1.1 —— we shall use this apparantly weaker condition to illustrate how this could be used to prove that (iv) $\Rightarrow$ (i) in 1.1). Since $\alpha_h(x_1) = x_1$ for $h \in H$ and $x \in A_1^\alpha(u)$, it follows that $\{x(u, k)\}$ is a central sequence in $A \times_\beta H$. (But note that $x(u, k) \in M(A \times_\beta H)$.)

There is an H-covariant irreducible representation ρ of A. (See (i) $\Rightarrow$ (ii) in 1.1 for $\beta = \alpha | H$.) Then ρ induces a representation $\bar{\rho}$ of $A \times_\beta H$ on the same space. Hence the weak closure of $\bar{\rho}(A\lambda(h))$ contains 1 for any $h \in H$. This implies that there is a sequence $\{a(h, n)\}$ in A such that $\| a(h, n) \| = 1$ and $\{a(h, n)\lambda(h)\}$ is a central sequence in $A \times_\beta H$ and $\| x a(h, n) \lambda(h) \| \to \| x \|$ for any $x \in A \times_\beta H$.

Let $\{u_k\}$ be a dense sequence in the set of $u \in \mathcal{U}$ with $u_{11}(h) = 1$, $h \in H$ such that each isolated point in this set appears infinitely often in $\{u_k\}$. Let $\{I_k\}$ be a sequence of non-zero ideals of $A \times_\beta H$ such that for each non-zero ideal J of $A \times_\beta H$ there is a k with $J \supset I_k$. Let $\{h_k\}$ be a dense sequence in H such that each isolated point appears infinitely often in $\{h_k\}$. Let $\{b_k\}$ be a dense sequence in the unit ball of $A \times_\beta H$, and let $\{\varepsilon_k\}$ be a decreasing sequence of positive numbers such that $\varepsilon_1 < 1$ and $\lim \varepsilon_k = 0$.

Let $e_1 \in T \cap I_1$, $H(e_1) = \{x \in A \times_\beta H : ex = xe = x\}$ (as before) and $p_1 = p(e_1)$ the open projection corresponding to $H(e_1)$. Choose k_1 such that

$$\lambda_1 \equiv \| p_1(x(u_1, k_1) + x(u_1, k_1)^*)p_1 \| \geq \delta[u_1]$$

$$\| [b_1, x(u_1, k_1)] \| < \varepsilon_1$$

where $x(u_1, k_1)$ may be replaced by $\lambda x(u_1, k_1)$ with $\lambda \in \mathbb{T}$ to obtain the first inequality. Then choose $a_1 \in H(e_1) \cap T$ such that

$$\| y_1 \| = \sup \operatorname{Spec}(y_1) > \lambda_1 - \varepsilon_1 \delta[u_1]$$

where $y_1 = a_1 \{ x(u_1, k_1) + x(u_1, k_1)^* \} a_1$, by replacing $x(u_1, k_1)$ by $-x(u_1, k_1)$ if necessary. Define a continuous function f on $\mathbb{R}$ by

$$f(t) = \begin{cases} 1 & t \geq 2/3 \\ 3(t - 1/3) & 1/3 \leq t < 2/3 \\ 0 & t < 1/3. \end{cases}$$

Let $a_1' = f(y_1/(\varepsilon_1 \| y_1 \|) - \varepsilon_1^{-1} + 1)$ and choose $e_1' \in H(a_1') \cap T$. Since $e_1 y_1 = y_1$, it follows that $e_1 a_1' = a_1'$ and so $e_1 e_1' = e_1'$. Let $p_1' = p(e_1')$ and choose ℓ_1 such that

$$\lambda_1' \equiv \| p_1'(a(h_1, \ell_1)\lambda(h_1) + \lambda(h_1)^* a(h_1, \ell_1)^*)p_1' \| > 1$$

$$\| [b_1, a(h_1, \ell_1)\lambda(h_1)] \| < \varepsilon_1 .$$

Then choose $a''_1 \in H(e'_1) \cap T$ such that

$$\|y'_1\| = \sup \mathrm{Spec}(y'_1) > \lambda'_1 - \varepsilon_1$$

where $y'_1 = a''_1 \{a(h_1, \ell_1)\lambda(h_1) + \lambda(h_1)^* a(h_1, \ell_1)^*\} a''_1$, by replacing $a(h_1, \ell_1)$ by $-a(h_1, \ell_1)$ if necessary. Let $a'''_1 = f(y'/(\varepsilon_1 \|y'_1\|) - \varepsilon_1^{-1} + 1)$ and choose $e_2 \in H(a'''_1) \cap T \cap I_2$. Then $e'_1 a''_1 = a''_1$ and $e'_1 e_2 = e_2$.

We repeat this procedure. Eventually we obtain a decreasing sequence $\{e_n\}$ and others satisfying appropriate conditions.

Let φ be a pure state of $A \times_\beta H$ such that $\varphi(e_n) = 1$ for all n, and we assert that π_φ satisfies the desired properties.

Since $e_n \in I_n$ and so $\|\varphi | I_n\| = 1$, π_φ is faithful.

Since $e_n a'_n = e_n$ and so $\varphi(a'_n) = 1$, it follows that

$$\varphi(y_n) \geq (1 - \varepsilon_n/3)\|y_n\| \geq (1 - \varepsilon_n/3)(\lambda_n - \varepsilon_n \delta_{[u_n]})$$

where $y_n = a_n\{x(u_n, k_n) + x(u_n, k_n)^*\} a_n$ and $a'_n = f(y_n/(\varepsilon_n \|y_n\|) - \varepsilon_n^{-1} + 1)$. On the other hand it follows that

$$\varphi(y_n) \leq \| p_n \{x(u_n, k_n) + x(u_n, k_n)^*\} p_n \| \varphi(a_n^2) = \lambda_n \varphi(a_n^2).$$

Thus for a subsequence $\{n_m\}$ with $[u_{n_m}] = \gamma$, $\varphi(a_{n_m}^2)$ converges to 1 and so

$$\lim_m \varphi((1 - a_{n_m})^2) = 0.$$

Hence it follows that

$$\liminf_m \mathrm{Re}\, \varphi(x(u_{n_m}, k_{n_m})) \geq \delta_\gamma/2.$$

Since $e_{n+1} a'''_n = e_{n+1}$ and so $\varphi(a'''_n) = 1$, it follows that

$$\varphi(y'_n) \geq (1 - \varepsilon_n/3)\|y'_n\| \geq (1 - \varepsilon_n/3)(\lambda'_n - \varepsilon_n)$$

where $y'_n = a''_n\{a(h_n, \ell_n)\lambda(h_n) + \lambda(h_n)^* a(h_n, \ell_n)^*\} a''_n$, and $a'''_n = f(y'_n/(\varepsilon_n \|y'_n\|) - \varepsilon_n^{-1} + 1)$. On the other hand it follows that

$$\varphi(y_n') \leq \lambda_n' \varphi(a_n'^2).$$

Since $\lambda_n' > 1$, it follows that $\varphi(a_n'^2)$ converges to 1 as $n \to \infty$. Thus we obtain that

$$\liminf \operatorname{Re} \varphi(a(h_n, \ell_n)\lambda(h_n)) = 1.$$

From this property it easily follows that $\pi_\varphi^{**}|A$ is irreducible. (Because for any $h \in H$, choose a subsequence $\{n_m\}$ such that $h_{n_m} \longrightarrow h$. Let Q be a weak limit point of $\pi_\varphi^{**}(a(h_{n_m}, \ell_{n_m}))$ and then it follows that $Q\,\pi_\varphi^{**}(\lambda(h)) = 1$ or $\pi_\varphi^{**}(\lambda(h)) \in \pi_\varphi^{**}(A)''$, which implies that $\pi_\varphi(A \times_\beta H)'' = \pi_\varphi^{**}(A)''$.)

Let $\rho = \pi_\varphi^{**}|A$ and define a representation $\tilde{\rho}$ of A on $L^2(G, H_\varphi)$ by

$$\tilde{\rho} = \int_G^{\oplus} \rho \circ \alpha_s \, ds.$$

Let Z be the center of $\tilde{\rho}(A)''$. Let $u \in \mathcal{U}$ be such that $u_{11}(h) = 1$, $h \in H$. Choose a subsequence $\{n_m\}$ such that $[u_{n_m}] = [u]$ and $u_{n_m} \to u$. Let $x^{(m)} \in A_1^\alpha(u_{n_m})$ such that $x_1^{(m)} = x(u_{n_m}, k_{n_m})$. Then $\{x_i^{(m)}\}$ is a central sequence in A for any $i = 1, 2, \ldots, d = \dim(u)$, and we may assume that

$$\lim_m \varphi(x_i^{(m)}) = \lambda_i$$

exists. Then $\delta_{[u]}/2 \leq \lambda_1 \leq 1$ and $|\lambda_i| \leq \sqrt{d}$, $i = 2, \ldots, d$ (which follows from the way x_i's are defined). Since

$$\varphi \circ \alpha_s(x_j^{(m)}) = \sum_{i=1}^{d} \varphi(x_i^{(m)}) u_{ij}(s),$$

it follows that for $j = 1, \ldots, d$,

$$\sum_{i=1}^{d} \lambda_i u_{ij} \in Z,$$

where we regard Z as a subalgebra of $L^\infty(G)$.

This argument shows the following: For any $u \in \mathcal{U}$ with $P_u^H \neq 0$ and any unit column vector $\xi \in \mathbb{C}^d$ there is a row vector $\lambda \in \mathbb{C}^d$ such that $\lambda\xi \neq 0$ and

$$(\lambda u)_j = \sum_{i=1}^{d} \lambda_i u_{ij} \in Z. \qquad (*)$$

Since Z is a right-translation invariant subalgebra of $L^\infty(G)$, there is a closed subgroup N of G such that $Z \cong L^\infty(N \backslash G)$. We have to show that $N = H$. Since $\rho \circ \alpha_h \sim \rho$ for $h \in H$, it follows that $H \subset N$ (see the proof of 2.2).

Let $s \in N$. Fix $u \in \mathcal{U}$ with $P_u^H \neq 0$ and let $d = \dim u$. For each unit vector $\xi \in \mathbb{C}^d$ with $P_u \xi = \xi$, one has a row vector $\tilde{\xi} \in \mathbb{C}^d$ such that $(\tilde{\xi} u)_j \in Z$ for $j = 1, \ldots, d$. Then it follows that

$$\sum \tilde{\xi}_i u_{ij}(hs) = \sum \tilde{\xi}_i u_{ij}(h), \quad h \in H.$$

This implies that $\tilde{\xi} P_u u(s) = \tilde{\xi} P_u$. Then, since $\tilde{\xi} P_u \xi = \tilde{\xi}\xi \neq 0$, and ξ is an arbitrary vector in $P_u \mathbb{C}^d$, one obtains that $P_u u(s) = P_u$. But note that $C(H \backslash G)$ is the closed linear span of $(P_u u)_{ij}$ with $u \in \mathcal{U}$. Since $(P_u u(s))_{ij} = (P_u)_{ij}$, Hs is not distinguishable from H in $H \backslash G$, i.e., $s \in H$.

Proof of (ii) $\Rightarrow$ (i) of 9.1. Suppose (ii) and let φ be a state as in (ii). Let $u \in \mathcal{U}$ with $P_u \neq 0$, and let $d = \dim u$. For any unit vector $\xi \in P_u \mathbb{C}^d$, it follows that

$$\langle u\xi, \xi \rangle \in \tilde{\pi}_\varphi(A)''$$

where $\langle u\xi, \xi\rangle$ is the multiplication operator on $L^2(G, H)$ defined by

$$(\langle u\xi, \xi\rangle \Phi)(t) = \langle u(t)\xi, \xi\rangle \Phi(t), \quad t \in G.$$

Then there is a central sequence $\{y^{(n)}\}$ in A such that $\| y^{(n)} \| \leq \| \langle u\xi, \xi\rangle \| = 1$ and

$$\tilde{\pi}(y^{(n)}) \longrightarrow \langle u\,\xi, \xi\rangle \qquad \text{strongly*}.$$

Let

$$y_{ij}^{(n)} = d\int_G u_{ij}(t)\,\alpha_{t^{-1}}(y^{(n)})\,dt\ .$$

Then $\alpha_s(\underset{\sim}{y}^{(n)}) = \underset{\sim}{y}^{(n)}u(s)$, where $\underset{\sim}{y}^{(n)}$ is the $d\times d$ matrix $(y_{ij}^{(n)})$ and $\alpha_s(\underset{\sim}{y}^{(n)}) = (\alpha_s(y_{ij}^{(n)}))$, and also

$$\pi_\varphi(y_{ij}^{(n)}) \longrightarrow \xi_i\,\bar{\xi}_j\ .$$

By using the method in Section 7 one can assume that $\|\underset{\sim}{y}^{(n)}\| \le 1$. Hence $\{\xi^*\underset{\sim}{y}^{(n)}\xi\}$ is a central sequence of elements of $A^{\alpha|H}$ with norm less than or equal to 1 and satisfies that

$$\pi_\varphi(\xi^*\underset{\sim}{y}^{(n)}\xi) \to 1.$$

Now for each $u\in\mathcal{U}$ with $P_u \neq 0$ and each unit vector $\xi\in P_u\mathbf{C}^d$ with $d = \dim u$, there is a central sequence $\{y_n\}$ in $\{x\xi\ :\ x\in A_1^\alpha(u),\ \|x\xi\| = 1\}$ such that

$$\lim \|ay_n\| = \|a\|, \qquad a\in A.$$

On the other hand for each $v\in\mathcal{U}(H)$, the set of irreducible unitary matrix representations of H, and each unit vector $\xi\in\mathbf{C}^d$ with $d = \dim v$, there is a central sequence $\{z_n\}$ in $\{x\xi\ :\ x\in A_1^{\alpha|H}(v),\ \|x\xi\| = 1\}$ such that

$$\lim \|az_n\| = \|a\|\ , \qquad a\in A.$$

We apply the procedure described in the proof of (i) $\Rightarrow$ (ii). What we obtain here is a pure state ψ of A satisfying

(a) π_ψ is faithful,

(b) for any $u\in\mathcal{U}$ with $P_u\neq 0$ and any unit vector $\xi\in P_u\mathbf{C}^d$ the weak closure of $\{\pi_\psi(x\xi)\ :\ x\in A_1^\alpha(u)\}$ contains 1, and

(c) for any $v\in\mathcal{U}(H)$ and any unit vector $\xi\in\mathbf{C}^d$ the weak closure of $\{\pi_\psi(x\xi)\ :\ x\in A_1^{\alpha|H}(v)\}$ contains 1.

From (c) it follows that $\pi_\psi | A^H$ is irreducible. Because for $v \in \mathcal{U}(H)$, $x \in A_1^{\alpha|H}(v)$ and any unit vector $\xi \in \mathbf{C}^d$ there is a row vector $\tilde{\xi} \in \mathbf{C}^d$ such that

$$\pi_\psi(x\tilde{\xi}^*) = \sum \pi_\psi(x_i)\overline{\tilde{\xi}_i} \in \pi_\psi(A^H)'', \qquad \tilde{\xi}\xi \neq 0.$$

Since the set of $\tilde{\xi}$ with $\xi \in \mathbf{C}^d$ spans $\mathbf{C}^d$, one obtains that $\pi_\psi(x_i) \in \pi_\psi(A^H)''$ for all i. This shows that $\pi_\psi | A^H$ is irreducible.

Now we proceed to the proof that $\pi_\psi | A^\alpha$ is irreducible. Let $u \in \mathcal{U}$ with $P_u \neq 0$, $x \in A_1^\alpha(u)$, and ξ a unit vector of $P_u\mathbf{C}^d$. Then there is a row vector $\tilde{\xi} \in \mathbf{C}^d$ such that

$$\pi_\psi(x\tilde{\xi}^*) = \sum \pi_\psi(x_i)\overline{\tilde{\xi}_i} \in \pi_\psi(A^\alpha)'', \qquad \tilde{\xi}\xi \neq 0.$$

For any $h \in H$, since $xu(h) \in A_1^\alpha(u(h)^*uu(h))$ and $P_{u(h)^*uu(h)} = P_u$, it also follows that

$$\pi_\psi(xu(h)\tilde{\xi}^*) \in \pi_\psi(A^\alpha)''.$$

In particular,

$$\pi_\psi(xP_u\tilde{\xi}^*) \in \pi_\psi(A^\alpha)''.$$

Note that $(P_u\tilde{\xi}^*)^*\xi = \tilde{\xi}P_u\xi = \tilde{\xi}\xi \neq 0$. The set of $P_u\tilde{\xi}^*$ with unit vectors $\xi \in P_u\mathbf{C}^d$ spans $P_u\mathbf{C}^d$. Hence one can conclude that $\pi_\psi(A^H) \subset \pi_\psi(A^\alpha)''$, completing the proof.

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